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Original Article

Variable Mass Dynamics of Celestial Bodies Revisited With Modification

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Abstract

Whittaker introduced variable mass dynamics in Celestial Mechanics obtaining as Jacobi –Lagrange counterpart a differential equation involving the moment of inertia, where time is taken as the independent variable. Assuming a mass variation law depending on the masses of the particles in motion and using their total mass as the independent variable in this paper, Lagrange's *general* equation of motion and the aforesaid equation of Whittaker are reduced to some relevant forms.

1. Introduction

In n-body problem of Celestial Mechanics, Whittaker[1] considered that the celestial particles are of point masses which are continuously varying according to some definite law and as such he modified Lagrange-Jacoby inequality or formula with his mass-variation law. He also proved that Lagrange's general equations of motion are also valid for mass variation of the particle.

The present paper is aimed at converting the foregoing variable- mass dynamics equations into those with the total mass as the independent variable.

2. Whittaker's variable- mass dynamics with mass as the independent variable.

Whittaker's [1] variable-mass dynamics equation is recalled as

$$\ddot{\emptyset} - (\mu\dot{\emptyset}) = 2T+V \quad (1)$$

$$2\emptyset = \sum_i m_i\vec{r}_i^{\,2} \quad (1.1)$$

Wherein is used the mass variation law

$$\dot{m}_i(t) = \mu(t)m_i \quad (2)$$

where $\mu(t) = a\ function\ of\ time\ t\ associated\ with\ mass\ m_i$.

The equation of energy[4] as the sum of kinetic (T) and potential (V)energies was derived with the key equation(2) and is recollected as

$$T+V = h-\int_0^t T(t)\mu(t)dt, \qquad h = constant \quad (2.1)$$

m_i= mass of the i th particle at time t.

$2\emptyset = moment\ of\ inertia\ of\ the\ particles\ at\ time\ t.$

$M=m_1+m_2+m_3+m_4$.............$+m_n$= the total mass of the particles as celestial bodies

$\bar{r}_i$=position vector of the ith particle at time t. Then

$$\dot{M}(t) = \mu(t)M(t) \quad (3)$$

With the dot sign as the derivative with respect to time t. The rest of the variables appearing in (1) have already been defined [1]. Equation (3) gives

$$\frac{dM}{M} = \mu(t)dt$$

Or , $\log(M/M_0) = \int_0^t \mu d\tau$

Or, the total mass at time t is given by

$$M = M_0 e^{\int_0^t \mu d\tau} \quad (4)$$

where M_0= the total mass of the celestial particles at time t=0.

In order to obtain Whittaker's mass- variation dynamics with the total mass as the independent variable the mass –variation law needs to be reformed as

$\dot{m}_i = \mu(M)m_i$; i= 1,2,3 ……………..n

so that

$$\dot{M}(t) = \mu(M)M(t) \quad (5)$$

$$t = \int_{M0}^{M} \frac{dM}{M.\mu(M)} \quad (6)$$

where $\mu(M)$

= a function of the total mass at time t. Now the solution(6) to equation (5)yields the time t explicitly interms of M.

Because of (5), equation (1) can be rewritten in the form

$$\frac{d}{dM}\left(\frac{d\emptyset}{dM}.\frac{dM}{dt}\right)\frac{dM}{dt} - \frac{d(\mu\emptyset)}{dM}.\frac{dM}{dt} = 2T+V$$

$$\text{Or,}\quad \mu M\frac{d^2\emptyset}{dM^2}+(M\frac{d\mu}{M})\frac{d\emptyset}{dM} - \emptyset\frac{d\mu}{dM} = \frac{2T+V}{\mu M} \quad (7)$$

If the mass- variation law(2) is such that $\mu(M) = \frac{K}{M}$ where K is Constant, equation (7) becomes

$$M^2\frac{d^2\emptyset}{dM^2} - M\frac{d\emptyset}{dM} + \emptyset = \frac{2T+V}{K^2}M^2 \quad (8)$$

Similarly taking

μ = KM where K is constant, equation (7)reduces to the form

$$M^2\frac{d^2\emptyset}{dt^2} + M\frac{d\emptyset}{dx} - \emptyset = \frac{2T+V}{(KM)^2} \quad (9)$$

It can be noted that employing the generalized law of gravitational force [1,4] instead of inverse-square law of force, equation(1) takes up the following form

$$\mu M\frac{d^2\emptyset}{dt^2} + (M\frac{d\mu}{dM})\frac{d\emptyset}{dM} - \emptyset\frac{d\mu}{dM} = \{2T+V-\sum\frac{\partial(V_{ij})}{\partial r_{ij}}\}/(\mu M) \quad (10)$$

3. Variable- mass dynamics equation without involving potential energy

The variable mass dynamics equation with time as the independent variable without involving the potential energy function has been derived by the present author[4] (SN Maitra) and is recapitulated with the help of (1),(2) and (2.1) as

$$\dddot{\emptyset} - (\mu\dot{\emptyset})^{\cdot\cdot} = \dot{T} - \mu T \quad (11)$$

(The dot sign indicates the derivative with respect to time t)

However, employing the relationship (2), equation (11) can be transformed into a form with the total mass M as the independent variable:

$$\frac{d}{dM}\left\{\mu M\frac{d}{dM}\left(\mu M\frac{d\emptyset}{dM}\right)\right\} - \frac{d\left\{\mu M\frac{d(\mu\emptyset)}{dM}\right\}}{dM} = M\frac{d\ (\frac{T}{M})}{dM}) \tag{12}$$

Which further reduces in consideration of the two special cases (i)
$\mu = \frac{K}{M}$ and (ii)
$\mu = KM$
as earlier, respectively, to the forms

$$\frac{d^3\emptyset}{dM^3} - \frac{d^2(\frac{\emptyset}{M})}{dM^2}) = \frac{M}{K^2}\frac{d\ (\frac{T}{M})}{dM} \tag{13}$$

and

$$M^3\frac{d^3\emptyset}{dM^3} + 5M^2\frac{d^2\emptyset}{dM^2} - 2M\frac{d\emptyset}{dM} - 2\emptyset = \frac{1}{K^2}\frac{d}{dM}(\frac{T}{M}) \tag{14}$$

4. Variable mass dynamics equation without involving kinetic energy

Variable- mass dynamics equation concerned with this section has already been obtained by the present author [4] but with time as the independent variable. The former equation is recalled as

$$\dddot{\emptyset}\text{-}(\mu\emptyset)\ddot{} + \mu\{\ddot{\emptyset}\text{-}(\mu\emptyset)\dot{}\} + \dot{V} + \mu V = 0 \tag{15}$$

which, by virtue of equation (2), can be rewritten having the total mass M as the independent variable:

$$\frac{d\{\mu M\frac{d}{dM}\left(\mu M\frac{d\emptyset}{dM}\right)\}}{dM} - \frac{d\{\mu M\frac{d(\mu M)}{dM}\}}{dM} + \mu\left\{\frac{d}{dM}\left(\mu M\frac{d\emptyset}{dM}\right) - \frac{d(\mu\emptyset)}{dM}\right\} + \frac{dV}{dM} + \frac{V}{M} = 0 \tag{16}$$

Now if $\mu M = K = constant, then$ (16) simplies to

$$K^2\frac{d^3\emptyset}{dM^3} - K^2\frac{d^2(\frac{\emptyset}{M})}{dM} + \frac{K^2}{M}\{\frac{d^2\emptyset}{dM^2} - \frac{d\left(\frac{\emptyset}{M}\right)}{dM}\} + \frac{dV}{dM} + \frac{V}{M} = 0$$

0r,

$$\frac{d^{3\emptyset}}{dM^3} + \frac{1}{M}\{\frac{d^2\emptyset}{dM^2} - M\frac{\mathrm{d}^2(\emptyset/\mathrm{M})}{\mathrm{dM}^2} - \frac{d(\emptyset/\mathrm{M})}{dM} + \frac{1}{K^2}\frac{d(VM)}{dM}\} = 0 \tag{17}$$

If $\mu = KM, equation$(16)becomes

$$\frac{d}{dM}\{M^2\frac{d(M^2\frac{d\emptyset}{dM})}{dM}\} - \frac{d}{dM}\left\{M^2\frac{d(M\emptyset)}{dM}\right\} + \frac{1}{M}\left\{\frac{d(M^2\frac{d\emptyset}{dM}}{dM}\right) - \frac{d(M\emptyset)}{dM} + \frac{d(VM)}{K^2dM}\} = 0 \tag{18}$$

5. Lagrange's equation with mass as the independent variable

Lagrange's dynamical equation can be recalled from any textbook of Classical Mechanics[5]:

$$\frac{d(\frac{\partial T}{\partial \dot{q_r}})}{dt} - \frac{\partial T}{\partial q_r} = Q_r \tag{19}$$

which, as shown by Whittaker, is also valid for variable masses whatsoever may be the mass- variation law. With the help of the present mass- variation law one can write

$$\dot{q_r} = q_r^1\mu M \tag{20}$$

where superscript 1 denotes the derivative with respect to the total mass M.

Using (20) in equation (19), because of (2)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_r}}\right) = \left\{\frac{d}{dM}\left(\frac{1}{\mu M}\frac{\partial T}{\partial q_r^1}\right)\right\}\mu M \qquad so\ that$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_r}}\right) - \frac{\partial T}{\partial q_r} = \frac{d}{dM}(\frac{\partial T}{\partial q_r^1}) - \frac{\partial T}{\partial q_r^1}\frac{d}{dM}\{\log(\mu M)\} - \frac{\partial T}{\partial q_r}$$

Hence Lagrange's equation(19) adopts the form

$$\frac{d}{dM}\left(\frac{\partial T}{\partial q_r^1}\right) - \frac{\partial T}{\partial q_r^1}\frac{d}{dM}\{\log(\mu M\} - \frac{\partial T}{\partial q_r} = Q_r(M) \tag{21}$$

If $\mu M = K\ constant, equation$ (21)reduces to

$$\frac{d}{dM}\left(\frac{\partial T}{\partial q_r^1}\right) - \frac{\partial T}{\partial q_r} = Q_r(M) \tag{22}$$

If $\mu = KM, equation\ (21) reduces\ to$

$$\frac{d}{dM}\left(\frac{\partial T}{\partial q_r^1}\right) - \frac{2}{M}\frac{\partial T}{q_r^1} - \frac{\partial T}{\partial q_r} = Q_r(M) \tag{23}$$

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